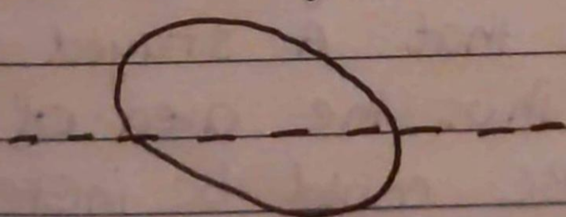
POTW Solution 8/29 - 9/5

The given problem is known as the isoperimetric problem.
I was initially about to give a mistaken proof of the 2D version using the calculus of variations but find the following geometric proof due to Steiner much more beautiful.

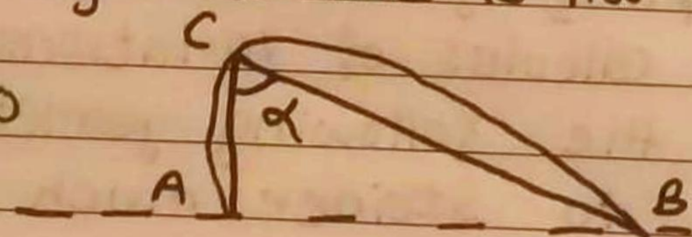
Take any figure with maximal area and divide it using a line such that the perimeters of both halves are equal.



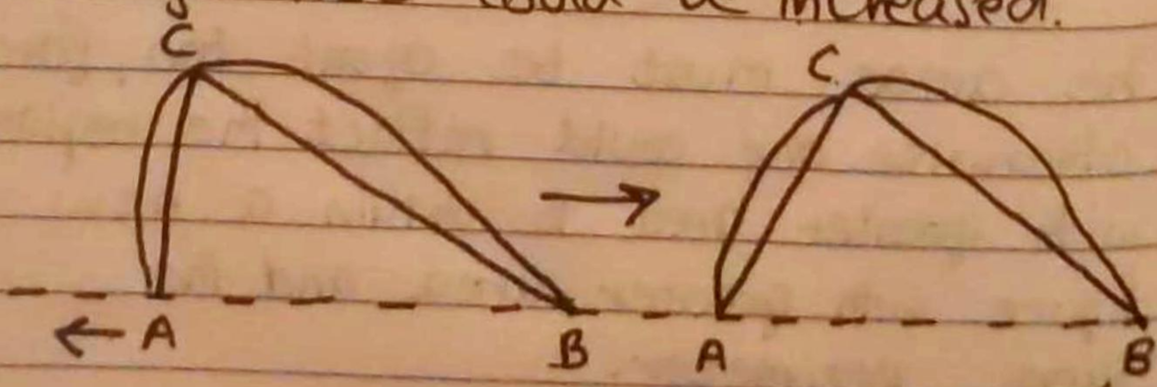
The areas must be equal too, for otherwise one could reflect the region with greater area to obtain a new figure with greater area and the same perimeter.

We focus on the top half.
Suppose it is not a semicircle.
Then there is a point on the boundary such that the angle subtended by the points cutting through our line is not 90°.

$\alpha < 90$



Now suppose the whole structure was rigid. with points B and C fixed but A along with the section of the figure resting on AC could move such that A stayed on the line AB. Thus the area of the triangle ABC could be increased.



Thus the area of our figure increases, ~~[illegible]~~ (maximum at $90^\circ = \alpha$) for the same perimeter.
Thus the angle subtended at each boundary point must be 90° and the shape must be a semi circle.
The same applies for the lower half.
q. e. d.

The corresponding generalisations to higher dimensions are considerably more difficult and can be found by looking up the isoperimetric problem.